Physics Factsheet

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Special Relativity

An inertial frame of reference is one in which a body remains at rest or moves with a constant velocity unless acted on by an unbalanced force, that is *a frame of reference in which Newton's First Law is obeyed.*

Quick Question No 1.
Two children are sitting on a bus facing each other. The bus does an emergency stop, and one child is forced onto the floor.
(a) Was the child on the floor facing the driver's cabin? Explain your answer.
(**b)** Could the bus be considered to be an inertial frame of reference?
(**c)** Give an example of a non-inertial frame of reference.

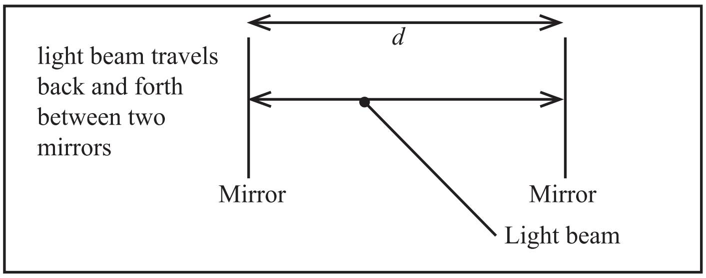
Answer to Quick Question No 1.
(a) Yes. He was facing in the direction of travel of the bus, and so carried on moving forwards when the bus (and seats) stopped (Newton's First Law). The other child was decelerated by the push of the back of his seat.
(b) The bus could *only be considered an inertial frame of reference* when *moving at constant velocity or when stationary.*
(c) An accelerating or decelerating frame of reference is non-inertial.

Albert Einstein used the concept of an inertial frame of reference and in his Theory of Special Relativity. This theory postulated that:
- ***The Laws of Physics, expressed in the form of equations, have the same form in all inertial frames of reference.*** This means that an experiment done in one inertial frame will give the same result in any other inertial frame.
- ***The speed of light in free space*** **always has the same value. Einstein said '*the speed of light is invariant in free space'.*** This means that measurements we make of the speed of light (in vacuo) are not affected by movement of the experimenter or of the light source.

Time Dilation
Proper time is defined as the time interval between two events when measured by an observer who is stationary with respect to the events. An observer moving at constant velocity relative to the events will measure a longer time.

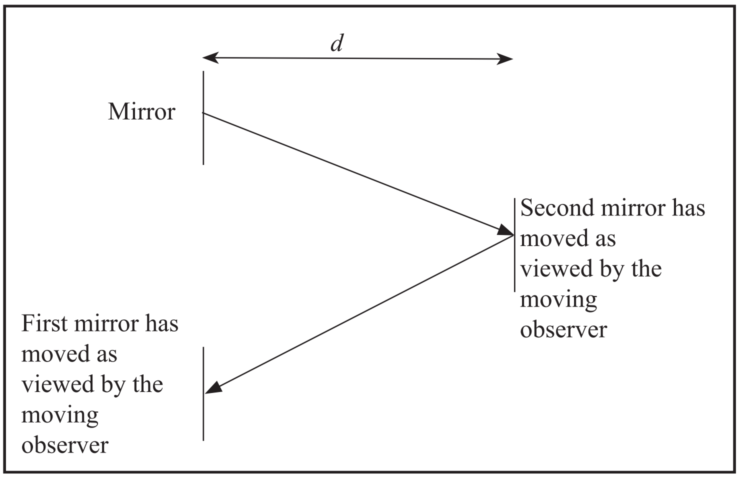
If you imagine a clock consisting of a beam of light reflecting between two mirrors with a stationary observer, the time, t, taken by the light to travel from one mirror and back, will be equal to the distance travelled, $2d$, divided by the velocity of light, c.

Figure 1 Time Dilation 1



Velocity of light, $c = 2d \div t$ and $t = 2d \div c$ where ***t* is the proper time**. However, if the observer is moving, the observer is in a different inertial frame from the clock. For this observer, the clock is moving with velocity v in the y direction, and so the distance travelled by the light increases.

Figure 2 Time Dilation 2



The distance travelled by the light wave is increased. If the clock is moving at v ms^{-1}, then in a time Δt taken for the light to travel from the first mirror to the second mirror and back, the first mirror has moved a distance $v\,\Delta t$. Using Pythagoras, we have distance travelled by the light $= 2\sqrt{\{d^2 + (v\,\Delta t \div 2)^2\}}$ ***The distance travelled is greater***.

The time observed to be taken is $\Delta t = 2\sqrt{\{d^2 + (v\,\Delta t \div 2)^2\}} \div c$
Given that the distance travelled is greater, the time observed to be taken will be greater (as c, the velocity of light, is the same for both but the distance travelled by the light is greater for the moving observer)

The time taken t for a particle moving at a speed *v* to undergo a process is given by

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In this equation t_0 is the time taken for a stationary particle to undergo the process.

($v = 0$ and so $\frac{v^2}{c^2} = 0$ and $t = t_0$).

At normal velocities $\frac{v^2}{c^2} << 1$ and so $\sqrt{1 - \frac{v^2}{c^2}}$ is effectively equal to one and the effect is negligible.

Evidence of Time Dilation from Muon Decay
Muons travel at very high speeds and evidence of time dilation has been found from muon decay. Muons are created when very high energy rays called cosmic rays reach the ionosphere from outer space. The muons are given a very high kinetic energy and so these muons travel at speeds very near the velocity of light (for example 0.996c).

Muons created on a mountain top were recorded at an observatory 2km below.

Muons have a half-life of 1.5 μs and it would be expected that in one half-life they would travel 0.996c × 1.5 μs (and so there would be 50% remaining after the corresponding distance). Travelling through a distance of 2000m would reduce their intensity by a factor of

$\left\{\frac{1}{2}\right\}^{\frac{2000}{450}} < \frac{1}{10}$ of their original intensity

However, measurements showed that approximately 80% of the muons reached the observatory. That is a much greater percentage than would be predicted by classical physics.

Practice Worked Calculations

Muons which have a half-life of 1.5 μs are created by cosmic radiation at the level of the top of a mountain and then travel at a velocity of 0.996c.

(a) Use a *non-relativistic calculation* to find the distance at which one half of the muons might be expected to remain.
After 1.5 μs = 1.5 × 10^{-6} s the muons would travel
0.996 × 3 × 10^{8} × 1.5 × 10^{-6} = 450m

(b) Use a *relativistic calculation* to find the distance at which one half of the muons would remain.

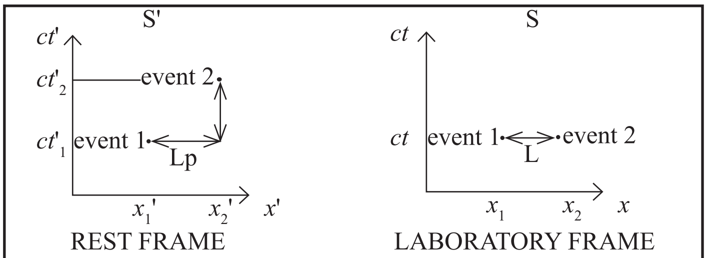
Using $t = \dfrac{t_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$

We have t_0 = 1.5 × 10^{-6} s v = 0.996 c
Putting these values into the formula gives
t = (1.5 × 10^{-6} s) ÷ (√{1 - [0.996^2 c^2 ÷ c^2]}) = 1.679 × 10^{-5} s and multiplying this by the speed at which the muons travel gives the distance travelled = 1.679 × 10^{-5} × 0.996 × 3 × 10^{8} = 5037 m

Length Contraction

If we consider a rod in an inertial frame of reference, the length of the rod is defined as the distance between its end-points at a given time measured in that inertial frame. The rod moves along an axis along which it is orientated, with a constant speed, *v*, and then the length of the rod can be related to two events that happen at the ends of the rod at the same time t. Event 1 at (t, x_1) and event 2 at (t, x_2), then the length of the rod at time t is equal to $x_2 - x_1$. Length of the rod = L = $x_2 - x_1$

Figure 3 Length Contraction



However, the same two events viewed in an inertial frame S' in which the rod is always at rest and is viewed along the *x*' axis occur at (t'_1, x'_1) and (t'_2, x'_2). These events may not occur at the same time, but because the rod is stationary and they will occur at the end points, the events will always be at x_1' and x_2'.

The length of the rod in its own rest frame called the proper length L_P will be given by $L_P = \Delta x' = x_2' - x_1'$

In the inertial frame S (laboratory frame) in which the rod moves along the *x* axis with a constant speed *v*, the length of the rod is related to two events which occur at the ends of the rod, x_1 and x_2, at the same time, *t*. This gives the length of the rod as $l = \Delta x = x_2 - x_1$.

The rod is shorter in the laboratory frame than in its own rest frame.

The length of an object moving with a speed *v* is reduced by a factor

$\sqrt{1-\dfrac{v^2}{c^2}}$ where *c* is the speed of light.

If *l* is the length of the moving object and l_0 is the length of the stationary object then $l = l_0\sqrt{1-\dfrac{v^2}{c^2}}$

For objects moving at normal velocities the fraction $\dfrac{v^2}{c^2}$ is <<1 and so the effect is negligible and the result $l = l_0$ *is obtained.*

The Equivalence of mass and energy

Einstein postulated that an object with rest mass, m_0 moving with a velocity, v has a relativistic mass, m given by

$m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$ where *c* is the speed of light in free space

As the speed increases and v^2 approaches the value of c^2, the value of the fraction $\dfrac{v^2}{c^2} \rightarrow 1$ and the denominator tends to zero that *m* tends to infinity

This means that no object which has mass can travel at speeds equal to, or greater than c, the speed of light. Particle accelerators increase the speeds of beams of fundamental particles. As they reach speeds approaching that of light, more and more energy must be supplied.

Einstein formulated the equivalence of energy and mass by extending his idea of relativistic mass, to give the equation $E = mc^2$
where the energy is in joules, the mass is in kg and *c* is in ms^{-1}

The total relativistic energy in joules of an object is given by

$m = \dfrac{m_0 c^2}{\sqrt{1-\dfrac{v^2}{c^2}}}$

where m_0 is the rest mass in kg of the object and *v* is its speed in ms^{-1}

Quick Question No 2. A fundamental particle is accelerated to 75% of the speed of light. Find its relativistic mass in terms of its rest mass, m_0

Answer to Quick Question No 2.

relativistic mass $= \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} = \dfrac{m_0}{\sqrt{1-\dfrac{0.75^2c^2}{c^2}}} = \dfrac{m_0}{\sqrt{1-0.75^2}} = \dfrac{m_0}{\sqrt{7 \div 16}} = \dfrac{4}{\sqrt{7}}m_0$

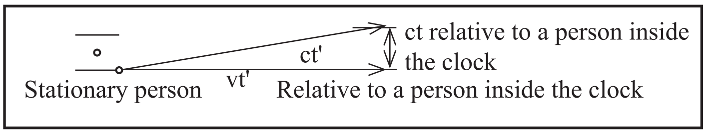
Time Dilation using Einstein's Clock

Time passed for a moving observer is less than time passed for a stationary observer. This means that when you move faster you age more slowly.

Einstein's Clock consists of two mirrors with a beam of light constantly bouncing up and down.

Time experienced by clock is ***t.***
Imagine that the clock moves to the right with velocity v

Figure 4 Time Dilation using Einstein's Clock



Time experienced by person is **t'**

The light takes a vertical path and relative to the stationary person, the light takes a diagonal path.

The time for the clock is t and the time for the person is ct'.
The right angled triangle formed gives the equation:
$(ct')^2 = (ct)^2 + (vt')^2$

Quick Question No 3. Rearrange the equation $(ct')^2 = (ct)^2 + (vt')^2$ to make t' the subject.

Answer to Quick Question No 3.

rearranging $\quad c^2(t')^2 = c^2t^2 + v^2(t')^2$

dividing by c^2 $\quad (t')^2 = t^2 + \frac{v^2}{c^2}(t')^2$

making t^2 the subject we have $\quad t^2 = (t')^2 - \frac{v^2}{c^2}(t')^2$

gathering terms in $(t')^2$ we have $\quad t^2 = (t')^2\left(1-\frac{v^2}{c^2}\right)$

taking the square root of each side gives us $t = t'\sqrt{\left(1-\frac{v^2}{c^2}\right)}$

and in terms of t' we have $\quad t' = \frac{t}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$

where t' is the stationary time, t is the moving time, v is the velocity of Einstein's clock and c is the velocity of light

Quick Question No 4. Use the equation for t' to explain why time dilation is usually ignored in calculations.

Answer to Quick Question No 4. The normal values for v are small compared to the value of c, hence for everyday situations $\frac{v^2}{c^2}$ is very small and so negligible and we have $t' = t$.

Practice Exam Style Questions

1. Muons of a certain type have a half- life of 1.5 microseconds and travel at 0.996c where c is the speed of light. Use the length contraction formula to show that after a distance of 2km approximately 24% of the muons will have decayed. (4 marks)

2. By definition, the unified mass unit, u is equal to one twelfth of the mass of a single carbon-12 atom. This gives u = 1.66 × 10^{-27} kg. Calculate the energy equivalent of a mass of 1u. (1 mark)

3. A rocket of mass m_0 is accelerated to 50% of the speed light. Find the increase in mass of the rocket. (1 mark)

4. A certain type of muon in its rest frame lives for Δt = 2.2 μs. The muon is travelling at a speed v = 3c/5 relative to an observer on Earth. Find the lifetime of the muon as measured by the observer. (2 marks)

Answers to Exam Style Questions

1. The equivalent distance in the muon frame of reference can be found using the length contraction formula

$$l = l_0\sqrt{1-\frac{v^2}{c^2}} = 177m$$

The distance travelled by a muon in one half life is 1.5μs × 0.996c = 448m. (1 mark). The intensity of the muons after travelling 448m or 177÷448 = 0.40 muon half-lives will be $I = \frac{I_0}{2^{0.295}}$

= 0.76 or 76%. (1 mark). Thus 24% have decayed. (1 mark)

2. $E = mc^2 = 1.66 \times 10^{-27}\,\text{kg} \times (3 \times 10^8\,\text{ms}^{-1})^2 = 1.49 \times 10^{-10}$ J. (1 mark)

3. $$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1-\frac{\left(\frac{1}{2}c\right)^2}{c^2}}} = \frac{m_0}{\sqrt{1-\frac{1}{4}}} = \frac{m_0}{\sqrt{\frac{3}{4}}} = \frac{2m_0}{\sqrt{3}} = 1.15m_0$$

So there is a 15% increase in the mass of the rocket when accelerated to 50% of the speed of light. (1 mark)

4. $$\sqrt{\left(1-\frac{v^2}{c^2}\right)} = \sqrt{1-\frac{\left(\frac{3c}{5}\right)^2}{c^2}} = \sqrt{1=\frac{9}{25}} = \sqrt{\frac{16}{25}} = 4/5$$ (1 mark)

And thus $t' = 5/4\ t$ and $\Delta t = 5/4 \times 2.2$ μs = 2.8 μs (1 mark)

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